11.1 Chi-Square Tests for Goodness of Fit

Introduction

Sometimes we want to examine the distribution of a single categorical variable in a population. The **chi-square goodness-of-fit test** allows us to determine whether a hypothesized distribution seems valid.

χ^2 GOF

We can decide whether the distribution of a categorical variable differs for two or more populations or treatments using a **chi-square test for homogeneity**.

We will often organize our data in a two-way table.

It is also possible to use the information in a two-way table to study the relationship between two categorical variables. The **chi-square test for independence** allows us to determine if there is convincing evidence of an association between the variables in the population at large.

May 2-11:07 AM

The Candy Man Can

Color	Blue	Orange	Green	Yellow	Red	Brown	Total
Count	9	8	12	15	10	6	60

The sample proportion of blue M&M's is $\hat{p} = \frac{9}{60} = 0.15$.

Since the company claims that 24% of all M&M' S® Milk Chocolate Candies are blue, we might believe that something fishy is going on. We could use the one-sample *z* test for a proportion to test the hypotheses

$$H_0: p = 0.24$$
$$H_a: p \neq 0.24$$

where *p* is the true population proportion of blue M&M' S®. We could then perform additional significance tests for each of the remaining colors.

Performing a one-sample *z* test for each proportion would be pretty inefficient and would lead to the problem of multiple comparisons.

May 2-11:16 AM

The Chi-Square Statistic

Performing one-sample *z* tests for each color wouldn't tell us how likely it is to get a random sample of 60 candies with a color distribution that differs as much from the one claimed by the company as this bag does (taking all the colors into consideration at one time).

For that, we need a new kind of significance test, called a **chi-square goodness-of-fit test.**

The null hypothesis in a chi-square goodness-of-fit test should state a claim about the distribution of a single categorical variable in the population of interest.

H_0: The company's stated color distribution for M&M'S® Milk Chocolate Candies is correct.

The alternative hypothesis in a chi-square goodness-of-fit test is that the categorical variable does *not* have the specified distribution.

H_a: The company's stated color distribution for M&M'S® Milk Chocolate Candies is not correct.

May 2-11:16 AM

The Chi-Square Statistic

We can also write the hypotheses in symbols as

H_0: $p_{blue} = 0.24$, $p_{orange} = 0.20$, $p_{green} = 0.16$,
$p_{yellow} = 0.14$, $p_{red} = 0.13$, $p_{brown} = 0.13$,

H_a: At least one of the p_i's is incorrect

where p_{color} = the true population proportion of M&M'S® Milk Chocolate Candies of that color.

The idea of the chi-square goodness-of-fit test is this: we compare the **observed counts** from our sample with the counts that would be expected if H_0 is true.

The more the observed counts differ from the **expected counts**, the more evidence we have against the null hypothesis.

May 2-11:16 AM

The Chi-Square Statistic

Assuming that the color distribution stated by Mars, Inc., is true, 24% of all M&M's® milk Chocolate Candies produced are blue.

For random samples of 60 candies, the average number of blue M&M's® should be (0.24)(60) = 14.40. This is our expected count of blue M&M's®.

Using this same method, we can find the expected counts for the other color categories:

Orange: (0.20)(60) = 12.00
Green: (0.16)(60) = 9.60
Yellow: (0.14)(60) = 8.40
Red: (0.13)(60) = 7.80
Brown: (0.13)(60) = 7.80

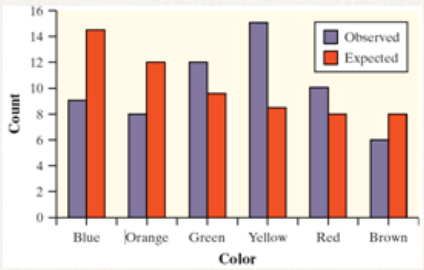
Color	Observed	Expected
Blue	9	14.40
Orange	8	12.00
Green	12	9.60
Yellow	15	8.40
Red	10	7.80
Brown	6	7.80

don't round!!

May 2-11:16 AM

The Chi-Square Statistic

To see if the data give convincing evidence against the null hypothesis, we compare the observed counts from our sample with the expected counts assuming H_0 is true. If the observed counts are far from the expected counts, that's the evidence we were seeking.



We see some fairly large differences between the observed and expected counts in several color categories. How likely is it that differences this large or larger would occur just by chance in random samples of size 60 from the population distribution claimed by Mars, Inc.?

To answer this question, we calculate a statistic that measures how far apart the observed and expected counts are. The statistic we use to make the comparison is the **chi-square statistic.**

May 2-11:17 AM

The Chi-Square Statistic

The **chi-square statistic** is a measure of how far the observed counts are from the expected counts. The formula for the statistic is

sum

$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$

where the sum is over all possible values of the categorical variable.

May 2-11:17 AM

The Chi-Square Statistic

The table shows the observed and expected counts for our sample of 60 M&M's® Milk Chocolate Candies. Calculate the chi-square statistic.

Color	Observed	Expected
Blue	9	14.40
Orange	8	12.00
Green	12	9.60
Yellow	15	8.40
Red	10	7.80
Brown	6	7.80

$$\chi^2 = \frac{(9-14.40)^2}{14.40} + \frac{(8-12.00)^2}{12.00} + \frac{(12-9.60)^2}{9.60} + \frac{(15-8.40)^2}{8.40} + \frac{(10-7.80)^2}{7.80} + \frac{(6-7.80)^2}{7.80}$$

$$\chi^2 = 2.025 + 1.333 + 0.600 + 5.186 + 0.621 + 0.415 = 10.180$$

df = 5

Think of χ^2 as a measure of the distance of the observed counts from the expected counts. Large values of χ^2 are stronger evidence against H_0 because they say that the observed counts are far from what we would expect if H_0 were true. Small values of χ^2 suggest that the data are consistent with the null hypothesis.

May 2-11:17 AM

A fair die?
Jenny made a six-sided die in her ceramics class and rolled it 60 times to test if each side was equally likely to show up on top.
Assuming that her die is fair, calculate the expected counts for each possible outcome.

Here are the results of 60 rolls of Jenny's ceramic die, along with the expected counts.

Calculate the chi-square statistic.

Outcome	Observed	Expected
1	13	10
2	11	10
3	6	10
4	12	10
5	10	10
6	8	10
Total	60	60

$\frac{1}{6}(60)$

$\chi^2 = \frac{(13-10)^2}{10} + \frac{(11-10)^2}{10} + \ldots + \frac{(8-10)^2}{10} = .9 + .1 + 1.6 + .4 + 0 + .4 = 3.4$

Apr 5-1:46 PM

CHECK YOUR UNDERSTANDING

Mars, Inc., reports that their M&M'S® Peanut Chocolate Candies are produced according to the following color distribution: 23% each of blue and orange, 15% each of green and yellow, and 12% each of red and brown. Joey bought a randomly selected bag of Peanut Chocolate Candies and counted the colors of the candies in his sample: 12 blue, 7 orange, 13 green, 4 yellow, 8 red, and 2 brown.

1. State appropriate hypotheses for testing the company's claim about the color distribution of M&M'S Peanut Chocolate Candies.

H_0: The company's claimed distribution of peanut m&m's is correct

H_a: The company's claimed distribution of peanut m&m's is not correct

2. Calculate the expected count for each color, assuming that the company's claim is true. Show your work.

Color	obs	Expected
Bl	12	.23(46) = 10.58
O	7	10.58
G	13	6.9
Y	4	6.9
R	8	5.52
Br	2	5.52

3. Calculate the chi-square statistic for Joey's sample. Show your work.

$\chi^2 = \frac{(12-10.58)^2}{10.58} + \frac{(7-10.58)^2}{10.58} + \frac{(13-6.9)^2}{6.9} + \ldots + \frac{(2-5.52)^2}{5.52} = 11.3724$

$df = 5$ $\quad .025 < pval < .05$

$\chi^2cdf(L, U, df) = \chi^2cdf(11.3724, 10000, 5) = .0445$

Apr 20-11:14 AM

The Chi-Square Distributions and *P*-Values

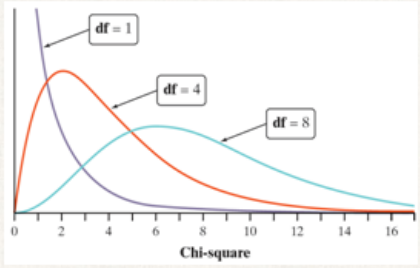
The sampling distribution of the chi-square statistic is not a Normal distribution. It is a right-skewed distribution that allows only positive values because χ^2 can never be negative.

large counts condition

When the expected counts are all at least 5, the sampling distribution of the χ^2 statistic is close to a **chi-square distribution** with degrees of freedom (df) equal to the number of categories minus 1.

The Chi-Square Distributions

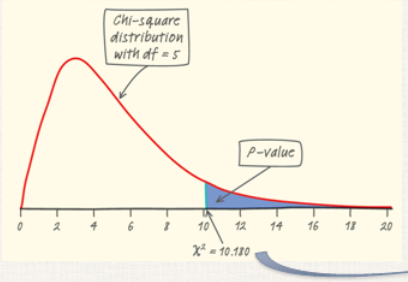
The chi-square distributions are a family of distributions that take only positive values and are skewed to the right. A particular chi-square distribution is specified by giving its degrees of freedom. The chi-square goodness-of-fit test uses the chi-square distribution with degrees of freedom = the number of categories - 1.



May 2-11:18 AM

The Chi-Square Distributions and *P*-Values

We computed the chi-square statistic for our sample of 60 M&M's to be $\chi^2 = 10.180$. Because all of the expected counts are at least 5, the χ^2 statistic will follow a chi-square distribution with df = 6 - 1 = 5 reasonably well when H_0 is true.



To find the *P*-value, use Table C and look in the df = 5 row.

	P		
df	.15	.10	.05
4	6.74	7.78	9.49
5	8.12	9.24	11.07
6	9.45	10.64	12.59

$\chi^2 = 10.18$
$df = 5$
$.05 \le pval \le .1$

distr
χ^2cdf
L U df
10.18, 10000, 5
pval = .07

Since our *P*-value is between 0.05 and 0.10, it is greater than $\alpha = 0.05$. Therefore, we fail to reject H_0. We don't have sufficient evidence to conclude that the company's claimed color distribution is incorrect.

When Jenny rolled her ceramic die 60 times and calculated the chi-square statistic, she got $\chi^2 = 3.4$. Using the appropriate degrees of freedom, calculate the *P*-value. What conclusion can you make about Jenny's die?

6 sides $\chi^2 = 3.4$
$df = 5$
$pval > .25$
χ^2cdf(3.4, 10000, 5) = .639

May 2-11:18 AM

Carrying Out a Test

Conditions for Performing a Chi-Square Test for Goodness of Fit

- **Random**: The data come a well-designed random sample or from a randomized experiment.
  - **10%**: When sampling without replacement, check that $n \leq (1/10)N$.
- **Large Counts**: All *expected* counts are greater than 5

Before we start using the chi-square goodness-of-fit test, we have two important cautions to offer.

•The chi-square test statistic compares observed and expected *counts*. Don't try to perform calculations with the observed and expected *proportions* in each category.

•When checking the Large Sample Size condition, be sure to examine the *expected* counts, not the observed counts.

May 2-11:18 AM

Carrying Out a Test

The Chi-Square Test for Goodness of Fit

Suppose the conditions are met. To determine whether a categorical variable has a specified distribution in the population of interest, expressed as the proportion of individuals falling into each possible category, perform a test of

H_0: The stated distribution of the categorical variable in the population of interest is correct.

H_a: The stated distribution of the categorical variable in the population of interest is not correct.

Start by finding the expected count for each category assuming that H_0 is true. Then calculate the chi-square statistic

$$\chi^2 = \sum \frac{(\text{Observed} - \text{Expected})^2}{\text{Expected}}$$

where the sum is over the k different categories. The P-value is the area to the right of χ^2 under the density curve of the chi-square distribution with $k-1$ degrees of freedom.

May 2-11:19 AM

Example: A test for equal proportions

Problem: In his book *Outliers*, Malcolm Gladwell suggests that a hockey player's birth month has a big influence on his chance to make it to the highest levels of the game. Specifically, since January 1 is the cut-off date for youth leagues in Canada (where many National Hockey League (NHL) players come from), players born in January will be competing against players up to 12 months younger. The older players tend to be bigger, stronger, and more coordinated and hence get more playing time, more coaching, and have a better chance of being successful.

To see if birth date is related to success (judged by whether a player makes it into the NHL), a random sample of 80 National Hockey League players from a recent season was selected and their birthdays were recorded.

May 2-11:19 AM

Example: A test for equal proportions

Problem: The one-way table below summarizes the data on birthdays for these 80 players:

Birthday	Jan – Mar	Apr – Jun	Jul – Sep	Oct – Dec
Number of Players	32	20	16	12

Do these data provide convincing evidence that the birthdays of all NHL players are evenly distributed among the four quarters of the year?

State: We want to perform a test of
H_0: The birthdays of all NHL players are evenly distributed among the four quarters of the year.
H_a: The birthdays of all NHL players are not evenly distributed among the four quarters of the year.

No significance level was specified, so we'll use $\alpha = 0.05$.

May 2-11:19 AM

Example: A test for equal proportions

Plan: If the conditions are met, we will perform a chi-square test for goodness of fit.

- Random: The data came from a random sample of NHL players.
  - 10%? Because we are sampling without replacement, there must be at least 10(80) = 800 NHL players. In the season when the data were collected, there were 879 NHL players.
- Large Counts: If birthdays are evenly distributed across the four quarters of the year, then the expected counts are all 80(1/4) = 20. These counts are all at least 5.

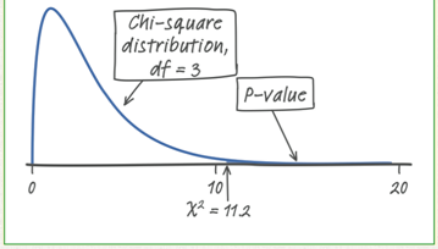
May 2-11:21 AM

Example: A test for equal proportions

Do: Test statistic

$$\chi^2 = \frac{(32-20)^2}{20} + \frac{(20-20)^2}{20} + \frac{(16-20)^2}{20} + \frac{(12-20)^2}{20}$$
$$= 7.2 + 0 + 0.8 + 3.2$$
$$= 11.2$$

As the excerpt shows, χ^2 corresponds to a P-value between 0.01 and 0.02.



	p		
df	**0.02**	**0.01**	**0.005**
2	7.82	9.21	10.60
3	9.84	11.34	12.84
4	11.67	13.28	14.86

Conclude:

Because the P-value, 0.011, is less than $\alpha = 0.05$, we reject H_0.

We have convincing evidence that the birthdays of NHL players are not evenly distributed across the four quarters of the year.

May 2-11:21 AM

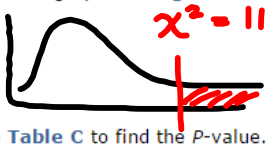
CHECK YOUR UNDERSTANDING

Let's continue our analysis of Joey's sample of M&M'S® Peanut Chocolate Candies from the previous Check Your Understanding (page 684).

1. Confirm that the expected counts are large enough to use a chi-square distribution. Which distribution (specify the degrees of freedom) should we use?

$df = 5$

2. Sketch a graph like Figure 11.4 on page 685 that shows the *P*-value.



3. Use Table C to find the *P*-value. Then use your calculator's χ^2 cdf command.

pval = .0445

4. What conclusion would you draw about the company's claimed color distribution for M&M'S® Peanut Chocolate Candies? Justify your answer.

pval $< \alpha = .05$ Reject H_0

There is convincing evidence that the color distribution of peanut m&m's is different than the company's claim

Apr 21-10:10 AM

Landline surveys

According to the 2000 census, of all U.S. residents aged 20 and older, 19.1% are in their 20s, 21.5% are in their 30s, 21.1% are in their 40s, 15.5% are in their 50s, and 22.8% are 60 and older. The table below shows the age distribution for a sample of U.S. residents aged 20 and older. Members of the sample were chosen by randomly dialing landline telephone numbers.

Category	Count	Expected
20–29	141	200.2
30–39	186	225.3
40–49	224	221.1
50–59	211	162.4
60+	286	238.9
Total	1048	

L1 L2

$df = 4$

Do these data provide convincing evidence that the age distribution of people who answer landline telephone surveys is not the same as the age distribution of all U.S. residents?

H_0: The age distribution of people who answer a landline is the same as the age dist. of US residents (based on 2000 census)

H_a: The age distribution of people who answer a landline is not the same as the age dist. of US residents (based on 2000 census)

10% ✓ > 10480 US residents 20 or older

random ✓ large counts ✓ all expected counts ≥ 5

χ^2GOF $df = 4$ $\chi^2 = 48.17$ pval ≈ 0

pval $< \alpha = .05$ Reject H_0

We have convincing evidence that the age dist of people who answer landline in NS is not the same as age dist of US residents

May 2-11:28 AM